Positive solution of the Legendre conjecture

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Abstract

In this paper, the strong form of the Legendre conjecture is shown. The logarithmic function is used to evaluate the minimum intervals between prime numbers ($\geq M^2$ and $\leq (M+1)^2$) .

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The Möbius function $\mu(n)$ is defined as follows.

$$\mu(n) := \begin{cases} 1 & \textit{product of even primes} \\ -1 & \textit{product of odd primes} \\ 0 & \textit{divided by the square of some prime number} \end{cases}$$

Let p_i be the ith prime number. $(p_1 = 2, p_2 = 3, \cdots)$ $[f(m)]$ represents a Gauss sign.

Theorem 1. $\{$*The minimum interval of the prime numbers between* M^2 *to* $(M+1)^2\} < \log M^2$.

Proof. Take $M^2 \leq m \leq (M+1)^2$.

$$\sum_{n;p_i \leq \sqrt{m}} [\frac{m}{n}]\mu(n)$$

be the sum of natural numbers have prime factors less than or equal to $\sqrt{m}$. Then

$$|\sum_{n;p_i \leq \sqrt{m}} [\frac{m}{n}]\mu(n)| = \#\{\textit{prime numbers less than } m \textit{ more than } \sqrt{m}\} + 1$$

holds. m' is the first prime number after m. m to m' does not contain a square number. Then

$$\sum_{n;p_i \leq M} [\frac{m'}{n}]\mu(n) - \sum_{n;p_i \leq M} [\frac{m}{n}]\mu(n) = 1$$

is got. Let K be the difference between

$$\sum_{n;p_i\le M}\frac{m'}{n}\mu(n)$$

and

$$\sum_{n;p_i\le M}\frac{m}{n}\mu(n)$$

$$\frac{1}{K}(\sum_{n;p_i\le M}\frac{m'}{n}\mu(n)-\sum_{n;p_i\le M}\frac{m}{n}\mu(n))=1$$

$$\frac{m'-m}{\frac{1}{K}(\sum_{n;p_i\le M}\frac{m'}{n}\mu(n)-\sum_{n;p_i\le M}\frac{m}{n}\mu(n))}$$

$$=\frac{m'-m}{\sum_{n;p_i\le M}[\frac{m'}{n}]\mu(n)-\sum_{n;p_i\le M}[\frac{m}{n}]\mu(n)}$$

This formula is

$$K\frac{1}{\sum_{n;p_i\le M}\frac{1}{n}\mu(n)}=m'-m$$

. The next result holds..

$$\sum_{n;p_i\le M}\frac{1}{n}\mu(n)=\Pi_{p_i\le M}(1-\frac{1}{p_i})$$

The previous formula is

$$K\Pi_{p_i\le M}(1-\frac{1}{p_i})^{-1}=m'-m$$

It is seen that about $\Pi_{p_i\le M}(1-\frac{1}{p_i})^{-1}$ correspond to one prime. So $\Pi_{p_i\le M}(1-\frac{1}{p_i})^{-1}$ is the average value of "inteval of primes".

The later part is achieved to the proof of the next result.

$$\Pi_{p_i\le\sqrt{m}}(1-\frac{1}{p_i})^{-1}\le\log m(for\ large\ m)$$

Let p_N be the last prime before $\sqrt{m}$.

$$\Pi_{p_i\ne p_N\le\sqrt{m}}(1-\frac{1}{p_i})^{-1}(1-\frac{1}{p_N})^{-1}-\Pi_{p_i\ne p_N\le\sqrt{m}}(1-\frac{1}{p_i})^{-1}$$

$$=\Pi_{p_i\ne p_N\le\sqrt{m}}(1-\frac{1}{p_i})^{-1}\frac{1}{p_N-1}$$

It is suposed inductively

$$\Pi_{p_i \le p_{N-1}}(1-\frac{1}{p_i})^{-1} \le \log p_{N-1}{}^2$$

Primally, in the case $\log p_{N-1} < p_N - p_{N-1}$,

$$\Pi_{p_i \le p_{N-1}}(1-\frac{1}{p_i})^{-1}\frac{1}{p_N - 1} \le \log p_{N-1}^2 \frac{1}{p_N - 1}$$

$$< 2(p_N - p_{N-1})\frac{1}{p_N - 1}$$

The value of this right-hand side is less than $2(\log p_N - \log p_{N-1})$.

In the case $\log p_{N-1} > p_N - p_{N-1}$, l is taken instead of 1. l satisfies that $\log p_{N-l} \ge p_N - p_{N-l+1}$ and $.\log p_{N-l} < p_N - p_{N-l}$.

$$\Pi_{p_i \le p_N}(1-\frac{1}{p_i})^{-1} - \Pi_{p_i \le p_{N-l}}(1-\frac{1}{p_i})^{-1} = \Pi_{p_i \le p_{N-l}}(1-\frac{1}{p_i})^{-1}\frac{1}{p_N - 1}$$

$$+O(\frac{1}{m}) < \log p_{N-l}^2 \frac{1}{p_N - 1} + O(\frac{1}{m}) < 2(p_N - p_{N-l})\frac{1}{p_N - 1} + O(\frac{1}{m})$$

p_{N-l} is taken instead of p_{N-1}

$$\Pi_{p_i \le \sqrt{m}}(1-\frac{1}{p_i})^{-1} \le \log m (for\ large\ m)$$

is got

By the constraction, for any $p_N - p_{N-l}$, abobe formula holds.

The prime number's long gap causes the defference of $\Pi_{p_i \le \sqrt{m}}(1-\frac{1}{p_i})^{-1}$ and $\log m$.

$M^2 \le p_\alpha, p_{\alpha-1} \le (M+1)^2$ give minimum interval of primes. K is taken as less than 1.

$$\Pi_{p_i \le M}(1-\frac{1}{p_i})^{-1} > p_\alpha - p_{\alpha-1}$$

Finally,

$$\log M^2 > p_\alpha - p_{\alpha-1}$$

is got. (The reason for K can be taken as less than 1 is that $(M+1)^2 - M^2 = 2M + 1 >> \log M^2$ (for large M^2). For finite M^2, It is calculated in the real example. For large M^2, "average value" can be used essentially.) □

By prime number theorem,

$$\frac{x}{\pi(x)} \sim \log x$$

holds. So theorem 1 is some kind of the strong evaluation.

References

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