Affirmative resolve of the Riemann Hypothesis

T.Nakashima
institute:none

Abstract

The Riemann Hypothesis is one of the propositions that has been stated without a proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse" (On the Number of Primes Less Than a Given Magnitude) ([1] B.Riemann) . The claim is that the real part of the non-trivial zero point of the zeta function would be 1/2. Proof has not been given for about 170 years. In this paper, we prove a little strong theorem of the proposition about the Mobius function equivalent to the Riemann Hypothesis.

1

Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known
Theorem

$$\sum_{n=1}^{m}\mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove next theorem. I call this proposition as S.R.H.(Strong Riemann Hypothesis)
Main Theorem(S.R.H.)

$$\sum_{n=1}^{m}\mu(n) = O(\sqrt{m}(\log\sqrt{m}+2))$$

$$S.R.H. \Leftrightarrow |\sum_{n=1}^{m}\mu(n)| \leq K\sqrt{m}(\log\sqrt{m}+2)(m \geq m_0)(\exists K, \exists m_0 > 0)$$

I prove the case $K \geq 1$, $m_0 = 2$. If this theorem holds, then

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}(\log\sqrt{m} + 2)) \Rightarrow \sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Rightarrow R.H$$

is got.

Lemma1.1

$$\sum_{n|m} \mu(n) = 1(m = 1)$$

$$\sum_{n|m} \mu(n) = 0(m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m} \mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let m's prime factorization be $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$. Then it becomes $\sum_{n|m} \mu(n) =_k C_0 -_k C_1 +_k C_2 -_k C_3 + \cdots _k C_k = (1-1)^k = 0$. □

Theorem1

$$\sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right] = 1$$

Proof. $\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$ is from Lemma1.1

$$\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+(\mu(1) + \mu(2) + \mu(4)) + \cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $\left[\frac{m}{2}\right]$ times that is a multiple of 2 less than m. $\mu(3)$ appears $\left[\frac{m}{3}\right]$ times that is a multiple of 3 less than m. In general, the number of occurrences of $\mu(n)(n < m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. That is

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+(\mu(1) + \mu(2) + \mu(4)) + \cdots = \sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right]$$

□

lemma2.1

$$If\ S.R.H.\ is\ true \Rightarrow$$

$$|\sum_{n=1}^{[\frac{m}{k+1}]}\mu(n)+\sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]}(\frac{m}{n}-[\frac{m}{n}])\mu(n)| \leq K\sqrt{\frac{m}{k}}(\log\sqrt{\frac{m}{k}}+2)(k=1,\cdots,m-1)$$

Proof. Start with $\sum_{n=1}^{[\frac{m}{k+1}]}\mu(n)$. Compare $\sum_{n=1}^{[\frac{m}{k+1}]}\mu(n)+\sum_{n>[\frac{m}{k+1}]}^{m'}(\frac{m}{n}-[\frac{m}{n}])\mu(n)$ and $\sum_{n=1}^{m'}\mu(n)$. It is assumed that m' is first increased from the original value. The former terms add up little by little. Therefore, a positive increase means that there were many positive terms. If negative terms and positive terms appear same time from first terms, then increase never occur. If only positive terms exists, this case is further easy. Therefore $\sum_{n=1}^{[\frac{m}{k+1}]}\mu(n)+\sum_{n>[\frac{m}{k+1}]}^{m'}(\frac{m}{n}-[\frac{m}{n}])\mu(n) < \sum_{n=1}^{m'}\mu(n)$.(The former is increase less than 1, the later is increased 1 or more. And the second increase means more difference of value.) If the minus is the same, the last $[\frac{m}{k}]$ item $|\sum_{n=1}^{[\frac{m}{k+1}]}\mu(n)+\sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]}(\frac{m}{n}-[\frac{m}{n}])\mu(n)| \leq K\sqrt{\frac{m}{k}}(\log\sqrt{\frac{m}{k}}+2)$. □

lemma2.2

$$-1 \leq |f(n)| \leq 1(n=1,\cdots,m) \Rightarrow \left|\sum_{n\leq m} f(n)\right| \leq m$$

Proof. I take sum of positive $f(n)(n \leq m)$ (represent F_1) and sum of the negative $f(n)(n \leq m)$(represent F_2). $|F_1|, |F_2| \leq m, |F_1+F_2| \leq m$ □

lemma2.3

$$2m^{\frac{1}{2}}-1 > 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{m}}$$

Proof.

$$2m^{\frac{1}{2}}-2 = \int_1^m x^{-\frac{1}{2}}dx > \frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{m}}$$

□

lemma2.4

$$2m^{\frac{1}{2}}-2 < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{m}}$$

Proof.

$$2m^{\frac{1}{2}} - 2 = \int_1^m x^{-\frac{1}{2}} dx < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}}$$

$\square$

Theorem2

$$If\ S.R.H\ is\ true \Rightarrow |\sum_{n=1}^{[m^{\frac{1}{2}}]} \frac{m}{n} \times \mu(n)| \leq 4K(2e^2 m^{\frac{1}{4}} + m^{\frac{1}{4}}(\log m^{\frac{1}{4}} \times m^{\frac{1}{8}} + 1))$$

Proof.

$$\sum_{n=1}^{[\sqrt{m}]} \frac{m}{n} \times \mu(n) = \sum_{n=1}^{[\sqrt{m}]} \frac{m}{n} \times \mu(n) - \sum_{n=1}^{[\sqrt{m}]} [\frac{m}{n}] \times \mu(n) + 1 = \sum_{n>[\frac{\sqrt{m}}{2}]}^{[\sqrt{m}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)$$

$$+ \sum_{n>[\frac{\sqrt{m}}{3}]}^{[\frac{\sqrt{m}}{2}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + \cdots + \sum_{n>[\frac{\sqrt{m}}{m^{\frac{1}{4}}}]}^{[\frac{\sqrt{m}}{m^{\frac{1}{4}}-1}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + \sum_{n=1}^{[m^{\frac{1}{4}}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + 1$$

$\frac{\sqrt{m}}{m^{\frac{1}{4}}-1} - m^{\frac{1}{4}} > 1$ is holds, and $m^{\frac{1}{4}} - \frac{\sqrt{m}}{m^{\frac{1}{4}}+1} < 1$ also holds. This means there are the empty terms. So, later the term correspond to $n \leq [m^{\frac{1}{4}}]$, formula's form is changed. By lemma2.1 $|\sum_{n=1}^{[\frac{\sqrt{m}}{2}]} \mu(n) + \sum_{n>[\frac{\sqrt{m}}{2}]}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| \leq Km^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2)$ Simillary $|\sum_{n=1}^{[\frac{\sqrt{m}}{3}]} \mu(n) + \sum_{n>[\frac{\sqrt{m}}{3}]}^{[\frac{\sqrt{m}}{2}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| \leq K\frac{m^{\frac{1}{4}}}{\sqrt{2}}$ $(\log \frac{m^{\frac{1}{4}}}{\sqrt{2}} + 2)$ You can continue this work. The terms correspond 1 to $[m^{\frac{1}{4}}]$, $|\sum_{n=1}^{[m^{\frac{1}{4}}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < [m^{\frac{1}{4}}]$ (by lemma2.2.) $|\sum_{n>[\frac{\sqrt{m}}{k+1}]}^{[\frac{\sqrt{m}}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| =$ $|\sum_{n=1}^{[\frac{\sqrt{m}}{k+1}]} \mu(n) + \sum_{n>[\frac{\sqrt{m}}{k+1}]}^{[\frac{\sqrt{m}}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) - \sum_{n=1}^{[\frac{\sqrt{m}}{k+1}]} \mu(n)| \leq K(\frac{m^{\frac{1}{4}}}{\sqrt{k}}(\log \frac{m^{\frac{1}{4}}}{\sqrt{k}} + 2) +$ $\frac{m^{\frac{1}{4}}}{\sqrt{k+1}}(\log \frac{m^{\frac{1}{4}}}{\sqrt{k+1}} + 2)) \leq 2\frac{m^{\frac{1}{4}}}{\sqrt{k}}(\log \frac{m^{\frac{1}{4}}}{\sqrt{k}} + 2)$. So, ($K$ is 1 or more.)

$$|\sum_{n=1}^{[\sqrt{m}]} \frac{\sqrt{m}}{n} \times \mu(n)| \leq 2K((1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{e^4}})m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2)$$

$$+(\frac{1}{\sqrt{e^4}+1} + \cdots + \frac{1}{(m^{\frac{1}{4}})^{\frac{1}{2}}})m^{\frac{1}{4}}(\log m^{\frac{1}{4}})) + [m^{\frac{1}{4}}] + 1 \leq 2K((1 + \frac{1}{\sqrt{2}} + \cdots +$$

$$\frac{1}{\sqrt{[e^4]}})m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2) + ([\frac{1}{[\sqrt{e^4}]+1} + \cdots + \frac{1}{([m^{\frac{1}{4}}])^{\frac{1}{2}}})m^{\frac{1}{4}}(\log m^{\frac{1}{4}}))$$

$+[m^{\frac{1}{4}}]+1 \leq 2K((2e^2-1)(m^{\frac{1}{4}}(\log m^{\frac{1}{4}}+2))+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times(2m^{\frac{1}{8}}-2-2e^2+2)+1))$

$$\leq 4K(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1))$$

By lemme 2.3 and lemma2.4, $2m^{\frac{1}{8}} > 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{[m^{\frac{1}{4}}]^{\frac{1}{2}}}$, $2e^2-1 > 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{[e^4]}}$, $2e^2-2 < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{[e^4]}}$. □

$\sum_{n=1}^{m}\mu(n)$ and $\sum_{n=1}^{m}(\frac{m}{n}-[\frac{m}{n}])\mu(n)$ each term is less than 1, and appears at the same time. So, roughly $\sum_{n=1}^{m}\mu(n) \approx \sum_{n=1}^{m}\frac{m}{n}\times\mu(n)-1$ would work. However, I could not prove it. I write it as "Conjecture" in below.

Conjecture

$$\sum_{n=1}^{m}\mu(n) \approx \sum_{n=1}^{m}\frac{m}{n}\times\mu(n)$$

lemma3.1

$$\sum_{n\leq\sqrt{m}}[\frac{\sqrt{m}}{n}] = \sum_{n\leq[\sqrt{m}]}[\frac{[\sqrt{m}]}{n}] = 1(m\ is\ natural\ number.)$$

Proof. $n = 2\cdots[\sqrt{m}]$ case, $[\sqrt{m}]-1+n \geq \sqrt{m} \Rightarrow [\frac{\sqrt{m}}{n}] = [\frac{[\sqrt{m}]}{n}]$, $n = 1$ case, $[\frac{\sqrt{m}}{1}] = [\frac{[\sqrt{m}]}{1}]$. By theorem1,$\sum_{n\leq[\sqrt{m}]}[\frac{[\sqrt{m}]}{n}] = 1$ □

lemma3.2

$$\log m + 1 > 1+\frac{1}{2}+\cdots+\frac{1}{m}$$

Proof.

$$\log m = \int_1^m \frac{1}{x}dx > \frac{1}{2}+\cdots+\frac{1}{m}$$

□

Theorem3

$$\sum_{n<m}\mu(n) = O(\sqrt{m}(\log\sqrt{m}+2))$$

I will prove that step by step.

Proof. Step1) Firstly use induction method. Assuming that "The absolute value of $\sum_{1\le n\le M}\mu(n)$ is (there exist fixed K,) $K\sqrt{M}\times(\log\sqrt{M}+2)$ against $2\le M<m$". I take K 1 or more. From theorem1

$$\sum_{n\le\sqrt{m}}\mu(n)\left[\frac{m}{n}\right]+\sum_{\sqrt{m}<n\le m}\mu(n)\left[\frac{m}{n}\right]=1$$

If I can use induction method, $\sum_{n\le\sqrt{m}}\mu(n)$ is the induction's $\sqrt{m}$ version. (I supose $m\ge 4$, $2\le m<4(\le e^{36})$ case, that is treated later.) Then, $\sum_{n\le\sqrt{m}}\mu(n)$'s size is less than $Km^{\frac{1}{4}}(\log m^{\frac{1}{4}}+2)$. By lemma2.2, Using $\sum_{n\le\sqrt{m}}\mu(n)[\frac{m}{n}]$ and $\sum_{n\le\sqrt{m}}\mu(n)\frac{m}{n}$. These are $[\sqrt{m}]$ terms, so the difference of size is less than $[\sqrt{m}]$. $\sum_{n\le\sqrt{m}}\mu(n)\frac{m}{n}=\sqrt{m}\sum_{n\le\sqrt{m}}\mu(n)\frac{\sqrt{m}}{n}=\sqrt{m}\times\frac{\sqrt{m}}{[\sqrt{m}]}\sum_{n\le\sqrt{m}}\mu(n)\frac{[\sqrt{m}]}{n}$. By Theorem2, the size is less than $4K\sqrt{m}\frac{\sqrt{m}}{[\sqrt{m}]}(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1))$ (Note: By lemma3.1, $\sum_{n\le\sqrt{m}}\mu(n)[\frac{m}{n}]\ne\sqrt{m}\sum_{n\le\sqrt{m}}[\frac{\sqrt{m}}{n}]$ $=\sqrt{m}$.) From above formula, confirm the two formula in below,

$$|\sum_{n\le\sqrt{m}}\mu(n)\left[\frac{m}{n}\right]|<4K\frac{m}{[\sqrt{m}]}(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1)) \qquad (1)$$

$$|\sum_{\sqrt{m}<n\le m}\mu(n)\left[\frac{m}{n}\right]|<4K\frac{m}{[\sqrt{m}]}(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1))+1 \qquad (2)$$

The following is obtained by calculation for (2). First term is sum of all terms satisfy $\left[\frac{m}{n}\right]=[\sqrt{m}]-1$, $m/\sqrt{m}=\sqrt{m}\ge[\sqrt{m}]$ and $m/(m/(\sqrt{m}-1))=\sqrt{m}-1\ge[\sqrt{m}-1]$, $(m/(m/(\sqrt{m}-1)+1)=m(\sqrt{m}-1)/(m+\sqrt{m}-1)<\sqrt{m}-1,)$ so the range is $\sqrt{m}$ to $m/(\sqrt{m}-1)$. Secondary term is sum of all terms satisfy $\left[\frac{m}{n}\right]=[\sqrt{m}]-2$, $m/(m/(\sqrt{m}-2))\ge[\sqrt{m}-2]$. The range is $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$. The last term satisfy $\left[\frac{m}{n}\right]=1$, that is $\frac{m}{2}$ to m.

$$\sum_{\sqrt{m}<n\le m}\mu(n)\left[\frac{m}{n}\right]=([\sqrt{m}]-1)\times\sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n)+([\sqrt{m}]-2)\times$$

$$\sum_{\sqrt{m}/(\sqrt{m}-1)<n\le m/(\sqrt{m}-2)}\mu(n)+\cdots+1\times\sum_{m/2<n\le m}\mu(n)$$

Step2) Transform from first term and the last term correspond $1\times$ by having the positive term element and the negative term element are moved to the left side.Noted that do not touch that $[\sqrt{m}]-1,[\sqrt{m}]-2\cdots,1$ in the transformation. Also, noted that use +1 and -1 at the same time. The left side's absolute value takes less than $4K\frac{m}{[\sqrt{m}]}(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1)+1$. $K\sqrt{m}\times\sqrt{m}>2.45\times K\times 4(\sqrt{m}+2)(2e^2m^{\frac{1}{4}}+m^{\frac{1}{4}}(\log m^{\frac{1}{4}}\times m^{\frac{1}{8}}+1))+1>$

$2.45 \times K \times 4\frac{m}{[\sqrt{m}]}(2e^2 m^{\frac{1}{4}} + m^{\frac{1}{4}}(\log m^{\frac{1}{4}} \times m^{\frac{1}{8}} + 1)) + 1 > \left|\sum_{\sqrt{m} \le n \le m} \mu(n) \left[\frac{m}{n}\right]\right|$ for $m > e^{36}$.$m \le e^{36}$ case treated later. ($\frac{m}{[\sqrt{m}]} < \sqrt{m} + 2 \Leftrightarrow (\sqrt{m} - [\sqrt{m}])(\sqrt{m} + [\sqrt{m}]) < 2\sqrt{m}$.) From first term to half term correspond to $[\frac{\sqrt{m}}{2}]$, the bias is $m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2) + \sqrt{2}m^{\frac{1}{4}}(\log \sqrt{2}m^{\frac{1}{4}} + 2)$. If the left side is positive then transform first terms and last term $1\times$ to the left side takes as possible as small absolute value. (The value is less than $(\frac{1}{2.45} - \frac{0.5+0.75}{2} \times 2 \times 0.25)m = (\frac{1}{2.45} - \frac{1.25}{4})m$.) From $[\frac{\sqrt{m}}{2}]$, 3 terms is taken at once. Here, From first term to the term correspond to $[\frac{\sqrt{m}}{2}] - 3$ The bias is $m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2) + \sqrt{\frac{2\sqrt{m}}{\sqrt{m}-6}}m^{\frac{1}{4}}(\log \sqrt{\frac{2\sqrt{m}}{\sqrt{m}-6}}m^{\frac{1}{4}} + 2)$. 3 terms transformed, If the new left side is positive, two terms takes as possible as positive and the last (3rd) term takes negative. This value is less than the rest terms' sum. First term to $[\frac{\sqrt{m}}{2}]$ is as first transformation. Next two terms only transformed that is less than $K\sqrt{m}$. Repeat this prosess, use $(\frac{1}{2.45} - \frac{1.25}{4})m < 0.5m$, If rest terms' sum is 0, except last term, all terms absolute value is less than $K\sqrt{m}$. The last term takes only bias, I can take this bias is less than $m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2) + \sqrt{m-1}(\log \sqrt{m-1} + 2) + 1$, and before terms absorb this bias. Put $|\sum_{n \le \sqrt{m}} \mu(n)|$ less than $\sqrt{m}$ into calculation. I can complete the induction method.
the last step) After transformation

$$left\ side = ([\sqrt{m}] - 1) \times \sum \mu(n) + ([\sqrt{m}] - 2) \times \sum \mu(n) + \cdots + 1 \times \sum \mu(n) n$$

All the absolute value of the right side is less than K times $\sqrt{m}$.

$$([\sqrt{m}] - 1) \times \sum \mu(n)$$

From here. (It might be 0.) $\sum \mu(n)$ is less than $\frac{1}{[\sqrt{m}]-1}K\sqrt{m}$. By lemma3.2

$$|\sum_{\sqrt{m} < n \le m} \mu(n)| = K(\sqrt{m})(\frac{1}{[\sqrt{m}]} + \frac{1}{[\sqrt{m}] - 1} + \cdots + 1) < K(\sqrt{m}(\log [\sqrt{m}] + 1))$$

$$|\sum_{n \le m} \mu(n)| = |\sum_{n \le \sqrt{m}} \mu(n) + \sum_{\sqrt{m} < n \le m} \mu(n)| \le K\sqrt{m}(\log[\sqrt{m}] + 2)$$

$|\sum_{n \le m} \mu(n)| < \sqrt{m}$, $(m < 10^{16})$ is true by [2], so $|\sum_{n \le m} \mu(n)| < \sqrt{m}(\log \sqrt{m} + 2)$, $(m < e^{36} < 10^{16})$ holds. $\square$

Special thanks: I was very grateful to my friend H. Tokitu for translating in English. I would like to express my gratitude to him.

References

[1] “ Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse ” ,B. Riemann Monatsberichte der Berliner Akademie. In Gesammelte Werke, Teubner, Leipzig (1892)

[2] Hurst, Greg (2016). ”Computations of the Mertens Function and Improved Bounds on the Mertens Conjecture”. arXiv:1610.08551 [math.NT].