Affirmative resolve of the Riemann Hypothesis

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Abstract

The Riemann Hypothesis is one of the propositions that has been stated without a proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse" (On the Number of Primes Less Than a Given Magnitude) ([1] B.Riemann) . The claim is that the real part of the non-trivial zero point of the zeta function would be 1/2. Proof has not been given for about 170 years. In this paper, we prove a little strong theorem of the proposition about the Mobius function equivalent to the Riemann Hypothesis.

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known
Theorem

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove next theorem. I call this proposition as S.R.H.(Strong Riemann Hypothesis)
Main Theorem(S.R.H.)

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}(\log \sqrt{m} + 2)) \Leftrightarrow \left| \sum_{n=1}^{m} \mu(n) \right| \le K\sqrt{m}(\log \sqrt{m} + 2)$$

$$(m \ge m_0)(\exists K, \exists m_0 > 0)$$

I prove the case $K \geq 1$, $m_0 = 2$. If this theorem holds, then

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}(\log \sqrt{m} + 2)) \Rightarrow \sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Rightarrow R.H$$

is got.

Lemma1.1

$$\sum_{n|m} \mu(n) = 1 (m = 1)$$

$$\sum_{n|m} \mu(n) = 0 (m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m} \mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let m's prime factorization be $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$. Then it becomes $\sum_{n|m} \mu(n) =_k C_0 -_k C_1 +_k C_2 -_k C_3 + \cdots_k C_k = (1-1)^k = 0$. $\square$

Theorem1

$$\sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right] = 1$$

Proof. $\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$ is from Lemma1.1

$$\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+ (\mu(1) + \mu(2) + \mu(4)) + \cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $\left[\frac{m}{2}\right]$ times that is a multiple of 2 less than m. $\mu(3)$ appears $\left[\frac{m}{3}\right]$ times that is a multiple of 3 less than m. $\mu(4)$ appears $\left[\frac{m}{4}\right]$ times which is a multiple of 4 less than m. In general, the number of occurrences of $\mu(n)(n < m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. That is

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+ (\mu(1) + \mu(2) + \mu(4)) + \cdots = \sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right]$$

It becomes. $\square$

lemma2.1

$$If\ S.R.H.\ is\ true \Rightarrow$$

$$\left|\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n) + \sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)\right| \le K\sqrt{\frac{m}{k}}(\log\sqrt{\frac{m}{k}}+2)(k=1,\cdots,m-1)$$

Proof. Start with $\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n)$. Compare $\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n) + \sum_{n>[\frac{m}{k+1}]}^{m'} (\frac{m}{n}-[\frac{m}{n}])\mu(n)$ and $\sum_{n=1}^{m'} \mu(n)$. It is assumed that m' is first increased from the original value. The former terms add up little by little. Therefore, a positive increase means that there were many positive terms. If negative terms and positive terms appear same time from first terms, then increase never occur. If only positive terms exists, this case is further easy. Therefore $\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n) + \sum_{n>[\frac{m}{k+1}]}^{m'} (\frac{m}{n} - [\frac{m}{n}])\mu(n) < \sum_{n=1}^{m'} \mu(n)$.(The former is increase less than 1, the later is increased 1 or more. And the second increase means more difference of value.) If the minus is the same, the last $[\frac{m}{k}]$ item $\left|\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n) + \sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)\right| \le K\sqrt{\frac{m}{k}}(\log\sqrt{\frac{m}{k}} + 2)$. □

lemma2.2

$$-1 \le |f(n)| \le 1(n=1,\cdots,m) \Rightarrow \left|\sum_{n\le m} f(n)\right| \le m$$

Proof. I take sum of positive $f(n)(n \le m)$ (represent F_1) and sum of the negative $f(n)(n \le m)$(represent F_2). $|F_1|,|F_2| \le m, |F_1 + F_2| \le m$ □

lemma2.3

$$2m^{\frac{1}{2}} > 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}}$$

Proof.

$$2m^{\frac{1}{2}} - 2 = \int_1^m x^{-\frac{1}{2}}dx > \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}}$$

□

Theorem2

$$If\ S.R.H\ is\ true \Rightarrow \left|\sum_{n=1}^{m} \frac{m}{n} \times \mu(n)\right| \le 4K(e^2(4\times e)+\sqrt{m}((\log\sqrt{m})\times m^{\frac{1}{4}}+1))$$

Proof.

$$\sum_{n=1}^{m} \frac{m}{n} \times \mu(n) = \sum_{n=1}^{m} \frac{m}{n} \times \mu(n) - \sum_{n=1}^{m} [\frac{m}{n}] \times \mu(n) + 1$$

$$= \sum_{n>[\frac{m}{2}]}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + \sum_{n>[\frac{m}{3}]}^{[\frac{m}{2}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + \cdots + \sum_{n>[\frac{m}{\sqrt{m}}]}^{[\frac{m}{\sqrt{m-1}}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)$$

$$+ \sum_{n=1}^{[\sqrt{m}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) + 1$$

$\frac{m}{\sqrt{m-1}} - \sqrt{m} > 1$ is holds, and $\sqrt{m} - \frac{m}{\sqrt{m+1}} < 1$ also holds. This means there are the empty terms. So, later the term correspond to $n \leq [\sqrt{m}]$, formula's form is changed. By lemma2.1 $\left| \sum_{n=1}^{[\frac{m}{2}]} \mu(n) + \sum_{n>[\frac{m}{2}]}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n) \right| \leq K\sqrt{m}$ $(\log \sqrt{m}$ $+ 2)$ Simillary $\left| \sum_{n=1}^{[\frac{m}{3}]} \mu(n) + \sum_{n>[\frac{m}{3}]}^{[\frac{m}{2}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) \right| \leq K\sqrt{\frac{m}{2}}(\log \sqrt{\frac{m}{2}} + 2)$ You can continue this work. The terms correspond 1 to $[m^{\frac{1}{2}}]$, $|\sum_{n=1}^{[\sqrt{m}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < [\sqrt{m}]$ (by lemma2.2.) $|\sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| = |\sum_{n=1}^{[\frac{m}{k+1}]} \mu(n) + \sum_{n>[\frac{m}{k+1}]}^{[\frac{m}{k}]} (\frac{m}{n} - [\frac{m}{n}])\mu(n) - \sum_{n=1}^{[\frac{m}{k+1}]} \mu(n)| \leq K(\sqrt{\frac{m}{k}} \log (\log \sqrt{\frac{m}{k}} + 2) + \sqrt{\frac{m}{k+1}}(\log \sqrt{\frac{m}{k+1}} + 2)) \leq 2K\sqrt{\frac{m}{k}}(\log \sqrt{\frac{m}{k}} + 2)$. So, ($K$ is taken as 1 or more.)

$$\left| \sum_{n=1}^{m} \frac{m}{n} \times \mu(n) \right| \leq 2K(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdots + \frac{1}{m^{\frac{1}{4}}})\sqrt{m}(\log \sqrt{m} + 2) + [m^{\frac{1}{2}}] + 1$$

$$\leq 2K(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdots + \frac{1}{([m^{\frac{1}{2}}])^{\frac{1}{2}}})\sqrt{m}(\log \sqrt{m} + 2) + [m^{\frac{1}{2}}] + 1 \leq 4K(e^2$$

$$(4 \times e) + \sqrt{m}((\log \sqrt{m}) \times m^{\frac{1}{4}} + 1))$$

Get. (I used lemma 2.3, $\int_{1}^{[m^{\frac{1}{2}}]} x^{-\frac{1}{2}} = 2([m^{\frac{1}{2}}])^{\frac{1}{2}} - 2 < 2m^{\frac{1}{4}} - 2$. □

$\sum_{n=1}^{m} \mu(n)$ and $\sum_{n=1}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n)$ each term is less than 1, and appears at the same time. So, roughly $\sum_{n=1}^{m} \mu(n) \approx \sum_{n=1}^{m} \frac{m}{n} \times \mu(n) - 1$ would work. However, I could not prove it. I write it as "Conjecture" in below.

Conjecture

$$\sum_{n=1}^{m} \mu(n) \approx \sum_{n=1}^{m} \frac{m}{n} \times \mu(n)$$

lemma3.1

$$\sum_{n \le \sqrt{m}} [\frac{\sqrt{m}}{n}] = \sum_{n \le [\sqrt{m}]} [\frac{[\sqrt{m}]}{n}] = 1 (m \ is \ natural \ number.)$$

Proof. $n = 2 \cdots [\sqrt{m}]$ case, $[\sqrt{m}] - 1 + n \ge \sqrt{m} \Rightarrow [\frac{\sqrt{m}}{n}] = [\frac{[\sqrt{m}]}{n}], n = 1$ case, $[\frac{\sqrt{m}}{1}] = [\frac{[\sqrt{m}]}{1}]$. By theorem1,$\sum_{n \le [\sqrt{m}]} [\frac{[\sqrt{m}]}{n}] = 1$ □

lemma3.2

$$\log m + 1 > 1 + \frac{1}{2} + \cdots + \frac{1}{m}$$

Proof.

$$\log m = \int_1^m \frac{1}{x} dx > \frac{1}{2} + \cdots + \frac{1}{m}$$

□

Theorem3

$$\sum_{n < m} \mu(n) = O(\sqrt{m} \log(m))$$

I will prove that step by step.

Proof. Step1) Firstly use induction method. Assuming that "The absolute value of $\sum_{1 \le n \le M} \mu(n)$ is (there exist fixed K,) $K\sqrt{M} \times (\log \sqrt{M} + 2)$ against $2 \le M < m$". I take K 1 or more. From theorem1

$$\sum_{n \le \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] + \sum_{\sqrt{m} < n \le m} \mu(n) \left[\frac{m}{n}\right] = 1$$

If I can use induction method, $\sum_{n \le \sqrt{m}} \mu(n)$ is the induction's $\sqrt{m}$ version. (I suppose $m \ge 4$, $2 \le m < 4 (\le e^{27})$ case, that is treated later.) Then, $\sum_{n \le \sqrt{m}} \mu(n)$'s size is less than $Km^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2)$. By lemma2.2, Using $\sum_{n \le \sqrt{m}} \mu(n)[\frac{m}{n}]$ and $\sum_{n \le \sqrt{m}} \mu(n)\frac{m}{n}$. These are $[\sqrt{m}]$ terms, so the difference of size is less than $[\sqrt{m}]$. $\sum_{n \le \sqrt{m}} \mu(n)\frac{m}{n} = \sqrt{m} \sum_{n \le \sqrt{m}} \mu(n)\frac{\sqrt{m}}{n} = \sqrt{m} \times \frac{\sqrt{m}}{[\sqrt{m}]} \sum_{n \le \sqrt{m}} \mu(n)\frac{[\sqrt{m}]}{n}$. By Theorem2, the size is less than $K\sqrt{m}\frac{\sqrt{m}}{[\sqrt{m}]} \times (4K(e(4 \times e^{\frac{1}{2}}) + m^{\frac{1}{4}}(\log m^{\frac{1}{4}}) \times m^{\frac{1}{8}} + 1))$ (Note: By lemma3.1, $\sum_{n \le \sqrt{m}} \mu(n)[\frac{m}{n}] \ne$

$\sqrt{m}\sum_{n\le\sqrt{m}}[\frac{\sqrt{m}}{n}]=\sqrt{m}$.) From above formula, confirm the two formula in below,

$$\left|\sum_{n\le\sqrt{m}}\mu(n)\left[\frac{m}{n}\right]\right|<K\frac{m}{[\sqrt{m}]}\times(e(4\times e^{\frac{1}{2}})+m^{\frac{1}{4}}(\log m^{\frac{1}{4}})\times m^{\frac{1}{8}}+1)\quad(1)$$

$$\left|\sum_{\sqrt{m}<n\le m}\mu(n)\left[\frac{m}{n}\right]\right|<K\frac{m}{[\sqrt{m}]}\times(e(4\times e^{\frac{1}{2}})+m^{\frac{1}{4}}(\log m^{\frac{1}{4}})\times m^{\frac{1}{8}}+1)+1\quad(2)$$

The following is obtained by calculation for (2). First term is sum of all terms satisfy $\left[\frac{m}{n}\right]=[\sqrt{m}]-1$, $m/\sqrt{m}=\sqrt{m}\ge[\sqrt{m}]$ and $m/(m/(\sqrt{m}-1))=\sqrt{m}-1\ge[\sqrt{m}-1]$, $(m/(m/(\sqrt{m}-1)+1)=m(\sqrt{m}-1)/(m+\sqrt{m}-1)<\sqrt{m}-1$,) so the range is $\sqrt{m}$ to $m/(\sqrt{m}-1)$. Secondary term is sum of all terms satisfy $\left[\frac{m}{n}\right]=[\sqrt{m}]-2$, $m/(m/(\sqrt{m}-2))\ge[\sqrt{m}-2]$. The range is $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$. The last term satisfy $\left[\frac{m}{n}\right]=1$, that is $\frac{m}{2}$ to m.

$$\sum_{\sqrt{m}<n\le m}\mu(n)\left[\frac{m}{n}\right]=([\sqrt{m}]-1)\times\sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n)+([\sqrt{m}]-2)\times$$

$$\sum_{\sqrt{m}/(\sqrt{m}-1)<n\le m/(\sqrt{m}-2)}\mu(n)+\cdots+1\times\sum_{m/2<n\le m}\mu(n)$$

Step2) Transform by having the positive term element and the negative term element are moved to the left side. Noted that do not touch that $[\sqrt{m}]-1,[\sqrt{m}]-2\cdots,1$ in the transformation. Also, noted that use $+1$ and -1 at the same time. Additionally, the left side's absolute value takes less than $K\frac{m}{[\sqrt{m}]}\times(e(4\times e^{\frac{1}{2}})+m^{\frac{1}{4}}(\log m^{\frac{1}{4}})\times m^{\frac{1}{8}}+1)+1$. The left side's number is not that important. The work will carry out following later. $K\sqrt{m}\times\sqrt{m}>K(\sqrt{m}+2)\times(e(4\times e^{\frac{1}{2}})+m^{\frac{1}{4}}(\log m^{\frac{1}{4}})\times m^{\frac{1}{8}}+1)+1>K\frac{m}{[\sqrt{m}]}\times(4m^{\frac{1}{4}}((\log m^{\frac{1}{4}}+2)m^{\frac{1}{8}}+1)+1)+1>\left|\sum_{\sqrt{m}\le n\le m}\mu(n)\left[\frac{m}{n}\right]\right|$ for $m>e^{27}$. $m\le e^{27}$ case treated later. ($\frac{m}{[\sqrt{m}]}<\sqrt{m}+2\Leftrightarrow(\sqrt{m}-[\sqrt{m}])(\sqrt{m}+[\sqrt{m}])<2\sqrt{m}$.) Eventually, all the right side of the absolute value is less than K times $\sqrt{m}$. Put $\left|\sum_{n\le\sqrt{m}}\mu(n)\right|$ less than $\sqrt{m}$ into calculation. I can complete the induction method.

the last step) After transformation

$$left\ side=([\sqrt{m}]-1)\times\sum\mu(n)+([\sqrt{m}]-2)\times\sum\mu(n)+\cdots+1\times\sum\mu(n)$$

All the absolute value of the right side is less than K times $\sqrt{m}$.

$$([\sqrt{m}]-1)\times\sum\mu(n)$$

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From here. (It might be 0.) $\sum \mu(n)$ is less than $\frac{1}{[\sqrt{m}]-1}K\sqrt{m}$. By lemma3.2

$$\left|\sum_{\sqrt{m}<n\leq m} \mu(n)\right| = K(\sqrt{m})(\frac{1}{[\sqrt{m}]}+\frac{1}{[\sqrt{m}]-1}+\cdots+1) < K(\sqrt{m}(\log[\sqrt{m}]+1))$$

$$\left|\sum_{n\leq m}\mu(n)\right| = \left|\sum_{n\leq\sqrt{m}}\mu(n) + \sum_{\sqrt{m}<n\leq m}\mu(n)\right| \leq K\sqrt{m}(\log[\sqrt{m}]+2)$$

$\log[\sqrt{m}] + 2 \leq \log\sqrt{m} + 2 \leq \frac{1}{2}\log m + \frac{1}{2}\log m = \log m$, for $m > e^4$.
$|\sum_{n\leq m}\mu(n)| < \sqrt{m}\log m, (2 \leq m \leq e^{27})$ is checked by computer. $\qquad\square$

Noted,I will mention that the transformation will be confirmed enough till the end in the theorem3's part2. Noted that K can be 1 (If conjecture3.3 is true.) and use induction method you will get $|\sum_{n=1}^{m'}\mu(n)| < \sqrt{m'}(\log\sqrt{m'} + 2)(m' < m)$ by S.R.H. This evaluation means Mobius function takes "average" value for any $m' < m$. In the work of theorem3's step2, it become useless, the left side takes sufficiently positive, and the right side is all negative at first and these are lower than $-\sqrt{m}$ and the rest of these are positive till the end and and all over $\sqrt{m}$. It is the same when the left side is negative. These cases can not occur. For example, he bias of positive and negative is $m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2)$ near $\sqrt{m}$, and the bias is $\sqrt{2}m^{\frac{1}{4}}(\log 2m^{\frac{1}{4}} + 2)$ near $2\sqrt{m}$. $2\sqrt{m}$ corresponds just half of terms. There is $[\frac{\sqrt{m}}{2}]$ pieces.($m^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 2) + \sqrt{2}m^{\frac{1}{4}}(\log 2m^{\frac{1}{4}} + 2) << [\frac{\sqrt{m}}{2}]$ The first terms takes enough positive and negative. These terms transformed with some terms that biased positive or negative, the left side does not satisfy "useless case" condition. (Like this, I use Mobius function's "average" property as the terms take positive and negative enough and the terms do not take extremely large value.)The transformation will be confirmed enough till the end

References

[1] "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse",B. Riemann Monatsberichte der Berliner Akademie. In Gesammelte Werke, Teubner, Leipzig (1892)