Guessing that the Riemann Hypothesis is unprovable

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 170 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I try to solve the proposition about the Mobius function equivalent to the Riemann Hypothesis. In this paper, first of all, I start from the failure of one proof. I guess the independence (unproofability) of a proposition equivalent to the Riemann Hypothesis about the Mobius function. First, the non-trivial result (theorem 1) regarding the Mobius function is shown. The theorem 2 writes about the failure of the proof. Then I states in the conjecture 1 that the Riemann Hypothesis is unprovable (independent).

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Handles a proposition equivalent to the Riemann Hypothesis. The Riemann Hypothesis is expressed as $R.H.$ $\mu(n)$ is the Mobius function.

The following theorem is well known.

theorem .

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

This paper deals with this equation on the left.

Lemma 1. *Let $\gamma_1 = 14.13\cdots$ be the smallest of the positive imaginary parts of the non-trivial zero-point of the zeta function. Let a be an arbitrary real number. In this condition,*

$$M_a(x) := \begin{cases} 0 \ for \ x < 0 \\ \sum_{n \leq x} \mu(n) n^{-a} - \frac{1}{\zeta(a)} \ for \ x \geq 0 \end{cases}$$

Lets the upper limit of the number of times that the sign is changed below Y be $V(M_a, Y)$

$$\varliminf_{Y \to \infty} \frac{V(M_a, Y)}{\log Y} \geq \frac{\gamma_1}{\pi}$$

Proof. ([10] Corollary4). □

I use this lemma for$\sum_{n=1}^{x} \frac{1}{n}\mu(n)$ and $\sum_{n=1}^{x} \mu(n)(x \leq m)$. These functions have sign changes more than $\log m \times \frac{\gamma_1}{\pi} \approx 4.5 \log m$ times. This number increases for large m. This value is about 31 for $m = 1000$. Actually I counted it $\sum_{n=1}^{x} \mu(n)(x \leq m)$ up to $m = 1000$, It changed the sign 35 times. It shows the accuracy of this lemma.

Furthermore, according to the original paper ([10]), there is a sign change point included in $[Y^{1-\epsilon'}, Y]$ for $Y > Y_0(\epsilon'), \epsilon' > 0$. In other words, it can be seen that there are enough points to change the sign.

Theorem 1.

$$\sum_{n=1}^{m} \mu(n) \ (t.c. \ \sum_{n=1}^{m} \frac{m}{n} \mu(n))$$

Here, t.c. means that it is too close.

Proof. m_0 is the last that satisfies $\sum_{n \leq m_0} \mu(n) = 0$ before m, and m' is the first to satisfy $\sum_{n \leq m'} \mu(n) = 0$ after m. By lemma 1, $\sum_{n \leq m} \frac{1}{n}\mu(n)$ and $\sum_{n \leq m} \mu(n)$ have enough change sign points. When $\sum_{n \leq m_0} \mu(n) = 0$, the coefficient changes gemtle. So I get $\sum_{n \leq m_0} \frac{1}{n}\mu(n)$ $(t.c.\ 0)$. (According to the example written later, at $m_0 = 920$ m_0 satisjies $\sum_{n=1}^{920} \mu(n) = 0$ It means something like $\sum_{n=1}^{920} \frac{920}{n}\mu(n) \risingdotseq 2.00191$ $(t.c.\ 0)$.) Especially in this case, it is necessary to take a larger scale. $\sum_{n \leq m} \frac{1}{n}\mu(n)$ changes frequent the sign. Therefore it can be $\frac{K}{m_0} > |\sum_{n \leq m_0} \frac{1}{n}\mu(n)|$. Since the form of the final term to be added is $\frac{1}{m_0}\mu(m_0)$, This formula is more accurate. In short, it is $0 = \sum_{n \leq m_0} \mu(n)$ $(t.c.\ \sum_{n \leq m_0} \frac{m_0}{n}\mu(n))$. If you proceed from m_0 $\sum_{n \leq m} \frac{m}{n}\mu(n)$ is $\sum_{n \leq \bar{m}-1} \frac{1}{n}\mu(n) + \mu(\bar{m})$ changes, $\sum_{n \leq m} \mu(n)$ changes $\mu(\bar{m}), \bar{m} \leq m$. (This is less than the amount of the difference between $\sum_{n \leq m'} \mu(n) = 0$ and $\sum_{n \leq m'} \frac{m'}{n}\mu(n)$ (this is very little).) Therefore, these two equations are roughly equal. (Too close means that the absolute value of $\sum_{n\le m_0} \frac{m_0}{n}\mu(n)$ is less than the absoliute value of $\sum_{n\le m}\mu(n)$ or $\sum_{n\le m}\frac{m}{n}\mu(n), m_0 \le m \le m'$ for almost all cases.) $\square$

example: m = 1000

$$\sum_{n=1}^{1000}\mu(n) = 2$$

$$\sum_{n=1}^{1000}\frac{1000}{n}\mu(n) \fallingdotseq 4.411$$

$m_0 = 920$

$$\sum_{n=1}^{920}\mu(n) = 0$$

$$\sum_{n=1}^{920}\frac{920}{n}\mu(n) \fallingdotseq 2.00191$$

$$\sum_{m=1}^{920}\sum_{n=1}^{m-1}\frac{1}{n}\mu(n) \fallingdotseq 2.00191$$

$m' = 1002$

$$\sum_{n=1}^{1002}\mu(n) = 0$$

$$\sum_{n=1}^{1002}\frac{1002}{n}\mu(n) \fallingdotseq 2.41969$$

example: m = 10000

$$\sum_{n=1}^{10000}\mu(n) = -23$$

$$\sum_{n=1}^{10000}\frac{10000}{n}\mu(n) \fallingdotseq -20.827$$

$m_0 = 9256$

$$\sum_{n=1}^{9256}\mu(n) = 0$$

$$\sum_{n=1}^{9256}\frac{9256}{n}\mu(n) \fallingdotseq 3.62119$$

$$\sum_{m=1}^{9256}\sum_{n=1}^{m-1}\frac{1}{n}\mu(n) \fallingdotseq 3.62119$$

$m' = 11117$

$$\sum_{n=1}^{11117}\mu(n) = 0$$

$$\sum_{n=1}^{11117}\frac{11117}{n}\mu(n) \fallingdotseq 0.414323$$

These results means $\sum_{n=1}^{m}\mu(n) < \sum_{n=1}^{m}\frac{m}{n}\mu(n)$.
example: m = 100000

$$\sum_{n=1}^{100000}\mu(n) = -48$$

$$\sum_{n=1}^{100000}\frac{100000}{n}\mu(n) \fallingdotseq -48.7228$$

example: m = 1000000

$$\sum_{n=1}^{1000000}\mu(n) = 212$$

$$\sum_{n=1}^{1000000}\frac{1000000}{n}\mu(n) \fallingdotseq 200.605$$

example: m = 10000000

$$\sum_{n=1}^{10000000}\mu(n) = 1037$$

$$\sum_{n=1}^{10000000}\frac{10000000}{n}\mu(n) \fallingdotseq 1015.24$$

These results means $\sum_{n=1}^{m}\mu(n) > \sum_{n=1}^{m}\frac{m}{n}\mu(n)$. The common result is $\sum_{n=1}^{m}\mu(n) \approx \sum_{n=1}^{m}\frac{m}{n}\mu(n)$
I saw like in this theorem 1, I haven't figured out why $\sum_{n=1}^{m}\mu(n)$ and $\sum_{n=1}^{m}\frac{m}{n}\mu(n)$ values are close for a long time. The answer is exactly $\sum_{n=1}^{m_0}\mu(n) = 0$ and $\sum_{n=1}^{m_0}\frac{m_0}{n}\mu(n)$ are close. The difference between the two expressions are caused by $|\sum_{n=1}^{m_0}\frac{m_0}{n}\mu(n) - \sum_{n=1}^{m_0}\mu(n)|$, the amount of change is different a little, but the two equations are almost same at m. This is the reason why the values of the two equations are close.

Theorem 2. *R.H. cannot be proved by my method.*

Proof. I start from

$$\sum_{\tilde{m}=1}^{m}\sum_{n=1}^{\tilde{m}-1}\frac{1}{n}\mu(n)\ (t.c.\ 0)$$

(Theorem 1). For $m_0 \leq M < m$ $\sum_{\tilde{m}=M}^{m}\sum_{n=1}^{\tilde{m}-1}\frac{1}{n}\mu(n)$, M is enough close to m, it is almost equal to $\frac{m-M}{m}$ times of $\sum_{n=1}^{m}\frac{m}{n}\mu(n)$. The following 2 patterns are established in the process of proof, so I can not decide which of these is correct. I can't prove it.
Pattern A When $\sum_{n=1}^{m}\frac{m}{n}\mu(n) > 0$

$$\sum_{n=1}^{m}\frac{m}{n}\mu(n) < Km^{\frac{1}{2}+\epsilon} \tag{1}$$

in the case of.

$$\sum_{n=1}^{m}\frac{1}{n}\mu(n) < Km^{-\frac{1}{2}+\epsilon} \tag{2}$$

Is true.

$$\int_{M}^{m} Kx^{-\frac{1}{2}+\epsilon}dx = 2Km^{\frac{1}{2}+\epsilon} - 2KM^{\frac{1}{2}+\epsilon}$$

As an approximate expression $(2Km^{\frac{1}{2}+\epsilon} - 2KM^{\frac{1}{2}+\epsilon})/(\sum_{n=1}^{m}\frac{m}{n}\mu(n)) \geq (2Km^{\frac{1}{2}+\epsilon} - 2KM^{\frac{1}{2}+\epsilon})/Km^{\frac{1}{2}+\epsilon} \approx \frac{m-M}{m}$ holds.

$$2Km^{\frac{1}{2}+\epsilon} - 2KM^{\frac{1}{2}+\epsilon} \geq \frac{m-M}{m}\sum_{n=1}^{m}\frac{m}{n}\mu(n) \tag{3}$$

It is a characteristic of this pattern that (1), (2), (3) hold, and these do not contradict each other.
Pattern B

$$\sum_{n=1}^{m}\frac{m}{n}\mu(n) \geq Km^{\frac{1}{2}+\epsilon} \tag{4}$$

in the case of.

$$\sum_{n=1}^{m}\frac{1}{n}\mu(n) \geq Km^{-\frac{1}{2}+\epsilon} \tag{5}$$

Is true.

$$2Km^{\frac{1}{2}+\epsilon} - 2KM^{\frac{1}{2}+\epsilon} \leq \frac{m-M}{m}\sum_{n=1}^{m}\frac{m}{n}\mu(n) \tag{6}$$

The characteristic of this pattern is that (4), (5), (6) hold. Pattern A and Pattern B completely stalled the proof. If there is hope, these two patterns may be a model of so-called "independence".

Let's take a look at $\sum_{n\leq m}\mu(n)$. $\sum_{\tilde{m}=1}^{m}\sum_{n=1}^{\tilde{m}-1}\frac{1}{n}\mu(n)$ has a negative effect on the calculation.

$$|-\sum_{\tilde{m}=1}^{m}\sum_{n=1}^{\tilde{m}-1}\frac{1}{n}\mu(n)+\sum_{n=1}^{m}\frac{m}{n}\mu(n)| < \ or\ \geq Km^{\frac{1}{2}+\epsilon}$$

$$|\sum_{n\leq m}\mu(n)| < \ or\ \geq Km^{\frac{1}{2}+\epsilon}$$

got. $\square$

To use it later, let us develop the so-called "non-standard analysis". We assume that you already know the basics of non-standard analysis. I take Frechet filter

$$\mathscr{F}_0=\{A\subset\mathbb{N}|\mathbb{N}\backslash A\ is\ a\ finite\ set\}$$

and let take the maximal filter ($\supset\mathscr{F}_0$) as the Ultra-filter. Write this as $\mathscr{F}$ and fix it hereafter.

$$\mathbb{R}^N=\{(a_1,a_2,a_3,\cdots)|a_i\in\mathbb{R}\}$$

$$(a_1,a_2,a_3,\cdots)\sim(b_1,b_2,b_3,\cdots)\Leftrightarrow\{k\in\mathbb{N}|a_k=b_k\}\in\mathscr{F}$$

I get $\mathbb{R}^N/\sim={}^*\mathbb{R}$. I want to handle hypernatural numbers and infinity ,so I consider ${}^*\mathbb{N}\subset{}^*\mathbb{R}$.

$$\infty_0:=[(1,2,3,\cdots)]\in{}^*\mathbb{N}$$

$$\infty_1:=[(K1^{\frac{1}{2}+\epsilon},K2^{\frac{1}{2}+\epsilon},K3^{\frac{1}{2}+\epsilon},\cdots)]\in{}^*\mathbb{R}$$

$$\infty_2:=[(2K1^{\frac{1}{2}+\epsilon},2K2^{\frac{1}{2}+\epsilon},2K3^{\frac{1}{2}+\epsilon},\cdots)]\in{}^*\mathbb{R}$$

The magnitude relation for hyperreal numbers is

$$[(a_1,a_2,a_3,\cdots)]\leq[(b_1,b_2,b_3,\cdots)]\Leftrightarrow\{k\in\mathbb{N}|a_k\leq b_k\}\in\mathscr{F}$$

and the natural number n is represented by

$$n=[(n,n,n,\cdots)]$$

Then

$$\forall n\leq\infty_0,\infty_1,\infty_2$$

Therefore, $\infty_0,\infty_1,\infty_2$ are all larger than any natural number, that is, satisfy the condition of infinity. With this premise, I make the following predictions.

conjecture 1. *The Riemann Hypothesis is unprovable. In other words, the Riemann Hypothesis is independent in the axiomatic system.*

In pattern A or pattern B, the composition is very beautiful. Therefore I guess that the Riemann Hypothesis is unprovable. This is sufficient considering that many people in the past have failed to prove and have been unresolved. It's a possible story. Let the whole axiomatic system be ZFC and pattern A be Φ_1. If pattern B contains even one example and is not empty, we will write it as $\Phi_2 = \neg\Phi_1$. In the range of natural numbers, I suppose Φ_1 is true in ZFC. Consider the hypernatural number here. This axiomatic system is represented as S. When the definition formula of the sum of the Mobius function is naturally extended as $\sum_{n\leq\infty}\mu(n)$ can take any fixed value. Let's take this value ∞_2, using our carefully prepared non-standard analysis here. In addition, let ∞_1 be the value of $Km^{\frac{1}{2}+\epsilon}$ at infinity. Then I get $\infty_1 \leq \infty_2$ at $m = \infty_0$. Φ_2 is true. At such times, the Riemann Hypothesis is unprovable and "independent" from theory. The reason why I used non-standard analysis is to convince myself that such a case certainly exists. A more intuitive explanation without using non-standard analysis. For example from $m = P - 3$ to $m = P$ if Φ_2 is true $\sum_{n=1}^{m}\frac{m}{n}\mu(n) \geq Km^{\frac{1}{2}+\epsilon}$ holds. I take $P \to \infty$. You can see $\infty = \infty(at\ \infty)$. (Pattern B seems to have disappeared cleanly. This defines $\sum_{n\leq N}\mu(n)$ as a fixed value at a certain N, there is no contradiction. This corresponds to the case where the sum of the Mobius functions is conveniently used. Of course, the sum of the Mobius functions that had shifted was not my starting point. It does not fit the formula equivalent to the Riemann Hypothesis. It is essential that ∞, which does not raise the issue, can be an example of Φ_2.) Of course, it is possible that the Riemann Hypothesis can be proved or disproved. However, it is quite possible that the Riemann Hypothesis is unprovable. I want to think that the Riemann Hypothesis is unprovable.

2 Other open issues

I will write about problems in which infinity is likely to be a problem, such as the Collatz conjecture and the Goldbach conjecture.

conjecture 2. *(Colatz conjecture)*
If the natural number n is odd, multiply it by 3 and add 1. If it is even, divide by 2. Then this calculation always goes to 1 *regardless of n.*

For this Collatz conjecture, consider the hypernatural numbers and consider the calculation when $n = \infty$. For example For $\infty_0 = [(1, 2, 3, \cdots)]$, I decide appropriately even or odd. Division and multiplication are well defined to function. If even, divide the terms that are divisible by 2 as much as they are divisible by 2, and if the numbers are odd, multiply the corresponding terms by 3 and add 1. See if it goes to $1 = [(1, 1, 1, \cdots)]$. In other words, whether the usual Collatz conjecture is correct or not equals to this situation.

conjecture 3. *(Goldbach conjecture)*
The double $2n$ of the natural number n can be written as the sum of two different prime numbers.

Consider hypernatural numbers for this Goldbach conjecture and consider the calculation when $n = \infty$. For example $2\infty_3 = [(4, 6, 8, \cdots)] = [(p_1, p_2, p_3, \cdots)] + [(q_1, q_2, q_3, \cdots)]$. This also reduces to the Goldbach conjecture for the general even $2n$.

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