Guessing that the Riemann Hypothesis is unprovable

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 163 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I guess the independence (unproofability) of a proposition equivalent to the Riemann Hypothesis about the Mobius function

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I define Moebius function $\mu(n)$ as

$$\mu(n) := \begin{cases} 1 & product\ of\ even\ primes \\ -1 & product\ of\ odd\ primes \\ 0 & divisible\ by\ the\ square\ of\ a\ prime\ number \end{cases}$$

The following theorem is well known.

theorem . *If the Riemann Hypothesis holds, then*

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

holds.

Proof. $M(x)$ is defined as

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=0.1}^{\infty} x^{-s} d(M(x))$$

$d(M(x))$ is the stiljes integral of $M(x)$.

$$= [M(x)x^{-s}]_{0.1}^{\infty} + s\int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

$$= s\int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

$\frac{1}{\zeta(s)}$ analytically continued up to finite. That this integral is finite at $Re(s) \neq 1/2$ equals that ζfunction takes no zero points at $Re(s) \neq 1/2$. Riemann Hypothesis is equivalent to $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$. □

If we add (or subtract) a large value at some N (denoted here as $f(N)$) instead of $\mu(N)$. Under condition $|f(n)| = N^{\delta+\epsilon'}$, (With the condition $Re(s) > \delta + \epsilon'$)

$$1/\zeta(s) = \lim_{N\to\infty} (\mu(1)/1^s + \mu(2)/2^s + \mu(3)/3^s + \cdots + f(N)/N^s)$$

holds, at $Re(s) = \delta$, $1/\zeta(s)$ diverges. . The sum of the Mobius functions at "infinity" can be determined arbitrarily to some extent.

Pattern A is the case

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

holds.

Pattern B is the case

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

does not hold..

To use it later, let us develop the so-called "non-standard analysis". We assume that you already know the basics of non-standard analysis. I take Frechet filter

$$\mathscr{F}_0 = \{A \subset \mathbb{N} | \mathbb{N} \backslash A \ is\ a\ finite\ set\}$$

and let take the maximal filter ($\supset \mathscr{F}_0$) as the Ultra-filter. Write this as $\mathscr{F}$ and fix it hereafter.

$$\mathbb{R}^N = \{(a_1, a_2, a_3, \cdots) | a_i \in \mathbb{R}\}$$

$$(a_1, a_2, a_3, \cdots) \sim (b_1, b_2, b_3, \cdots) \Leftrightarrow \{k \in \mathbb{N} | a_k = b_k\} \in \mathscr{F}$$

I get $\mathbb{R}^N / \sim =^* \mathbb{R}$. I want to handle hypernatural numbers and infinity ,so I consider ${}^*\mathbb{N} \subset^* \mathbb{R}$.

$$\infty_0 := [(1, 2, 3, \cdots)] \in^* \mathbb{N}$$

$$\infty_1 := [(K1^{\frac{1}{2}+\epsilon}, K2^{\frac{1}{2}+\epsilon}, K3^{\frac{1}{2}+\epsilon}, \cdots)] \in^* \mathbb{R}$$

$$\infty_2 := [(K1^{\frac{2}{3}}, K2^{\frac{2}{3}}, K3^{\frac{2}{3}}, \cdots)] \in^* \mathbb{R}$$

The magnitude relation for hyperreal numbers is

$$[(a_1, a_2, a_3, \cdots)] \leq [(b_1, b_2, b_3, \cdots)] \Leftrightarrow \{k \in \mathbb{N} | a_k \leq b_k\} \in \mathscr{F}$$

and the natural number n is represented by

$$n = [(n, n, n, \cdots)]$$

Then

$$\forall n \leq \infty_0, \infty_1, \infty_2$$

Therefore, $\infty_0, \infty_1, \infty_2$ are all larger than any natural number, that is, satisfy the condition of infinity. With this premise, I make the following predictions.

conjecture 1. *The Riemann Hypothesis is unprovable. In other words, the Riemann Hypothesis is independent in the axiomatic system.*

I guess that the Riemann Hypothesis is unprovable. This is sufficient considering that many people in the past have failed to prove and have been unresolved. It's a possible story. Let the whole axiomatic system be ZFC and pattern A be Φ_1. If pattern B contains even one example and is not empty, we will write it as $\Phi_2 = \neg\Phi_1$. In the range of natural numbers, I suppose Φ_1 is true in ZFC. Consider the hypernatural number here. This axiomatic system is represented as S. When the definition formula of the sum of the Mobius function is naturally extended as $|\sum_{n \leq \infty} \mu(n)|$ can take (some kind of limited) any fixed value. Let's take this value ∞_2, using our carefully prepared non-standard analysis here. In addition, let ∞_1 be the value of

$Km^{\frac{1}{2}+\epsilon}$ at infinity. Then I get $\infty_2 = O(\infty_1)$ is faulse at $m = \infty_0$. Φ_2 is true. At such times, the Riemann Hypothesis is unprovable and "independent" from theory. The reason why I used non-standard analysis is to convince myself that such a case certainly exists. A more intuitive explanation without using non-standard analysis. For example from $m = P - 3$ to $m = P$ if Φ_2 is true $|\sum_{n=1}^{m} \mu(n)| = O(m^{\frac{1}{2}+\epsilon})$ is faulse. I take $P \to \infty$. You can see $\infty = O(\infty)$ is faulse ($at\ \infty$). Of course, it is possible that the Riemann Hypothesis can be proved or disproved. However, it is quite possible that the Riemann Hypothesis is unprovable. I want to think that the Riemann Hypothesis is unprovable.

2 Other open issues

I will write about problems in which infinity is likely to be a problem, such as the Collatz conjecture and the Goldbach conjecture.

conjecture 2. *(Colatz conjecture)*
If the natural number n is odd, multiply it by 3 and add 1. If it is even, divide by 2. Then this calculation always goes to 1 regardless of n.

For this Collatz conjecture, consider the hypernatural numbers and consider the calculation when $n = \infty$. For example For $\infty_0 = [(1, 2, 3, \cdots)]$, Division and multiplication don't work unless you define them well. Divide by 2 for even elements, multiply by 3 and add 1 for odd elements. If the usual Collatz conjecture is correct, then all the elements finally come down to 1 or 2 or 4. And this is (by definition of Ultra-filter) $1 = [(1, 1, 1 \cdots)]$ or $2 = [(2, 2, 2 \cdots)]$ or $4 = [(4, 4, 4 \cdots)]$. Obviously, it will be the usual Collatz conjecture.

conjecture 3. *(Goldbach conjecture)*
The double $2n$ of the natural number n can be written as the sum of two different prime numbers.

Consider hypernatural numbers for this Goldbach conjecture and consider the calculation when $n = \infty$. For example $2\infty_3 = [(4, 6, 8, \cdots)] = [(p_1, p_2, p_3, \cdots)] + [(q_1, q_2, q_3, \cdots)]$. This also reduces to the Goldbach conjecture for the general even $2n$.

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