Affirmative resolve of Riemann Hypothesis

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Abstract

First, we prove the relation of the sum of the mobius function and Riemann Hypothesis. This relationship is well known. I prove next section, without any tool we prove Riemann Hypothesis about mobius function. This is very chalenging attempt.

1

We write R.H. as the omission of Riemann Hypothesis. $\mu(n)$ is the mobius function,

Theorem 1.1.

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}\log(m)) \Leftrightarrow R.H$$

proof. [1] We define $M(x)$ that is called Mertens function.

$$M(x) := \sum_{n=1}^{x} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=1}^{\infty} \frac{1}{x^s} d(M(x))$$

$d(M(x))$ is Stieltjes integral.

$$= [M(x)x^{-s}] + s \int_{x=1}^{\infty} M(x)x^{-s-1}dx$$

$M(x) < O(\sqrt{x}\log(x)) \Rightarrow$ This integral may not converge on $Re(s) = \frac{1}{2}$ and must converge on $Re(s) \neq \frac{1}{2}$

$\square$

2

We prove Riemann Hypothesis in this chapter.

Lemma 2.1.
$$\sum_{n|m} \mu(n) = 1(m = 1)$$

$$\sum_{n|m} \mu(n) = 0(m \neq 1)$$

proof. The case $m = 1$, $\sum_{n|m} \mu(n) = \mu(1) = 1$ is clear. $m \neq 1$, We factorise $m = p_1{}^{n_1}p_2{}^{n_2}p_3{}^{n_3}\cdots p_k{}^{n_k}$. We ignore zero term, $\sum_{n|m} \mu(n) = {}_kC_0 - {}_kC_1 + {}_kC_2 - {}_kC_3 + \cdots {}_kC_k = (1-1)^k = 0$. $\square$

[] is the Gauss sign.

Theorem 2.1.
$$\sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right] = 1$$

proof. By lemma2.1,$\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$.

$$\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1)+\mu(2)) + (\mu(1)+\mu(3)) + (\mu(1)+\mu(2)+\mu(4)) + \cdots$$

In this formula, we watch $\mu(n)$ as a character. $\mu(1)$ appears m times. $\mu(2)$ appears the number of the numbers of multiple of 2 lower than m that is $\left[\frac{m}{2}\right]$ times. $\mu(3)$ appears the number of the numbers of multiple of 3 lower than m that is $\left[\frac{m}{3}\right]$ times. $\mu(4)$ appears the number of the numbers of multiple of 4 lower than m that is $\left[\frac{m}{4}\right]$ times. Generally,in this formula, $\mu(n)(n \leq m)$

appears the number of the numbers of multiple of n lower than m that is $\left[\frac{m}{n}\right]$ times. So we get

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3)) + (\mu(1) + \mu(2) + \mu(4)) +$$

$$\cdots = \sum_{n \leq m} \mu(n) \left[\frac{m}{n}\right]$$

$\square$

Theorem 2.2.
$$\sum_{n \leq m} \mu(n) = O(\sqrt{m} \log(m))$$

proof. First,we use induction method.We assume "For $m_0 < M < m$,the absolute value of $\sum_{1 \leq n \leq M} \mu(n)$ is less than constant K-multiple of $\sqrt{M} \times \log \sqrt{M}$".
We take m_0 is big enohgh.
By Theorem2.1,

$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] + \sum_{\sqrt{m} < n \leq m} \mu(n) \left[\frac{m}{n}\right] = 1$$

More
$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{\sqrt{m}}{n}\right] \times \sqrt{m} = 1 \times \sqrt{m}$$

But
$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] \neq \sqrt{m}$$

So, we take α_0 a little larger than $\sqrt{m}$. We supose

$$\left| \sum_{n \leq \alpha_0} \mu(n) \left[\frac{m}{n}\right] \right| < \sqrt{m}$$

Why α_0 and $\sqrt{m}$ near ?
$$\sum_{n \leq \alpha_0} \mu(n) \frac{m}{n}$$

3

$$\sum_{n \leq \alpha_0} \mu(n) \left[\frac{m}{n}\right]$$

This two formulas' value is almost same. In some example, we see α_0.

$m = 10000$ case,

$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] = 311$$

$311/100$, so about 3 will be vanished at α_0.

In this case,

$$\sum_{n \leq \sqrt{m}} \mu(n) \times \sqrt{m} = 1 \times 100$$

$$\sum_{n \leq \sqrt{m}} \mu(n) \frac{m}{n} = 311.315 \cdots$$

$$\sum_{n \leq 103} \mu(n) \left[\frac{m}{n}\right] = 17$$

$$\sum_{n \leq \sqrt{m}} \mu(n) \times \sqrt{m} = 1 \times 100$$

Next calculation result is needed later.

$$\sum_{n \leq 103} \mu(n) = -2$$

But next result is very strong.

$$\sum_{n \leq 100} \mu(n) = 1$$

$m = 1000000$ case,

$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] = 4407$$

$4407/1000$, so about 4 will be vanished at α_0.

In this case

$$\sum_{n \leq \sqrt{m}} \mu(n) \times \sqrt{m} = 2 \times 1000$$

$$\sum_{n \leq \sqrt{m}} \mu(n) \frac{m}{n} = 4411.87 \cdots$$

$m = 100000000$ case,

$$\sum_{n \leq \sqrt{m}} \mu(n) \left[\frac{m}{n}\right] = -208247$$

$-208247/10000$, so about -20 will be vanished at α_0.
In this case,

$$\sum_{n \leq \sqrt{m}} \mu(n) \times \sqrt{m} = -23 \times 10000$$

$$\sum_{n \leq \sqrt{m}} \mu(n) \frac{m}{n} = -208269.976 \cdots$$

Next 2 formula is got.

$$\left|\sum_{n \leq \alpha_0} \mu(n) \left[\frac{m}{n}\right]\right| < \sqrt{m} \tag{1}$$

$$\left|\sum_{\alpha_0 < n \leq m} \mu(n) \left[\frac{m}{n}\right]\right| < \sqrt{m} - 1 \tag{2}$$

We do not use (1), we use

$$\left|\sum_{n \leq \alpha_0} \mu(n)\right| < \sqrt{m}$$

$$\sum_{\alpha_0 < n \leq m} \mu(n) \left[\frac{m}{n}\right] = (\left[\sqrt{m}\right] - 1) \times \sum_{\alpha_0 < n \leq m/\sqrt{m}-1} \mu(n) + (\left[\sqrt{m}\right] - 2) \times$$

$$\sum_{m/\sqrt{m}-1 < n \leq m/\sqrt{m}-2} \mu(n) + \cdots + 1 \times \sum_{m/2 < n \leq m} \mu(n)$$

We take i maximum value satisfies $\frac{m}{\sqrt{m}-i+1} < \frac{m}{\log^2 \sqrt{m}}$.We calculate well,$K \times$ ($\sqrt{m} - 1$) is larger than right side all term's order. plus term and minus term exist so we can do this. The element of plus term delete with coeficient either, and the element of minus term delete with coeficient either. But $(\sqrt{m} - 1), (\sqrt{m} - 2) \cdots, 1$ must not change. And 1 and -1 use same time. The baddest case, the result of cancelalation, the left side's absolute value is greater than $\sqrt{m} - 1$, we calculate right side minus (or plus) $\sqrt{m} - 1$. We calculate terms wuthout later i-term.We take first

term and other term plus minus cancelalation,the left side of this formula is gradually transeformed as it's absolute value is less than $\sqrt{m} - 1$.We continue the calculation. By assumption,$\left|\sum_{\sqrt{m}<n\leq\frac{m}{m-i+1}}\mu(n)\right|$'s absolute value is less than $K \times \sqrt{\frac{m}{m-i+1}}\log\sqrt{\frac{m}{m-i+1}} < K \times \sqrt{m}$.More, we take β_0 before i term. $\left|\sum_{1\leq n\leq\beta_0}\mu(n)[\frac{m}{n}]\right|$ is less than $\sqrt{m} - i + 1$.By α_0, β_0, the terms between α_0 and β_0 is all very small after cancelalation.So less than $K \times \sqrt{m}$.the terms between β_0 and m is all very small after cancelalation. By $\sum_{1<n\leq\beta_0}\mu(n)\left[\frac{m}{n}\right] + \sum_{\beta_0<n\leq m}\mu(n)\left[\frac{m}{n}\right] = 1$, so less than $K \times \sqrt{m}$,too All terms is less than $K \times \sqrt{m}$.

First induction is correct by later calculation.The statement is "For $m_0 < M \leq m$(specially $M = m$),the absolute value of $\sum_{1\leq n\leq M}\mu(n)$ is less than constant K-multiple of $\sqrt{M} \times \log\sqrt{M}$". Next order's property is important.If $f(x), g(x)$ is same sign, then

$$O(f(x) + g(x)) = O(max|f(x)|, |g(x|)$$

$$([\sqrt{m}] - 1) \times \sum\mu(n) + ([\sqrt{m}] - 2) \times \sum\mu(n) + \cdots + 1 \times \sum\mu(n)$$

This formula is right side of formula that already done to delete. First we calculate

$$([\sqrt{m}] - 1) \times \sum\mu(n)$$

(This term may be 0) $\sum\mu(n)$ has smaller order than $K \times (\sqrt{m} - 1)$ times $\frac{1}{([\sqrt{m}]-1)}$ Repeat similler argument, by $\log\sqrt{m} \approx \frac{1}{([\sqrt{m}])} + \frac{1}{([\sqrt{m}]-1)} + \cdots + 1$

$$\left|\sum_{\alpha_0<n\leq m}\mu(n)\right| < K((\sqrt{m} - 1) \times \log\sqrt{m})$$

(Here,We only caluculate one of plus term's sum and minus term's sum) Induction method is proved.

$$\sum_{n\leq m}\mu(n) = O(\sum_{n\leq\alpha_0}\mu(n) + \sum_{\alpha_0<n\leq m}\mu(n)) = O(\sqrt{m}\log(\sqrt{m})) = O(\sqrt{m}\log(m))$$

$\square$

Rifferences

[1]Riemann's zeta function,H.M.Edwards,Dover Books on Mathematics (28 Mar 2003)

[2]Uber die Anzahl der Primzahim unter einer gegebenen Grosse, Riemann. B, Monatsberichie der Berliner Akademie, November 1859.