Affirmative resolve of Riemann Hypothesis

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Abstract

First, we prove the relation of the sum of the mobius function and Riemann Hypothesis. This relationship is well known. I prove next section, without any tool we prove Riemann Hypothesis about mobius function. This is very chalenging attempt.

1

We write R.H. as the omission of Riemann Hypothesis. $\mu(n)$ is the mobius function,

Theorem 1.1.

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}\log(m)) \Leftrightarrow R.H$$

proof. [1] We define $M(x)$ that is called Mertens function.

$$M(x) := \sum_{n=1}^{x} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=1}^{\infty} \frac{1}{x^s} d(M(x))$$

$d(M(x))$ is Stieltjes integral.

$$= [M(x)x^{-s}] + s\int_{x=1}^{\infty} M(x)x^{-s-1}dx$$

$M(x) < O(\sqrt{x}\log(x)) \Rightarrow$ This integral may not converge on $Re(s) = \frac{1}{2}$ and must converge on $Re(s) \neq \frac{1}{2}$

□

2

We prove Riemann Hypothesis in this chapter.

Lemma 2.1.

$$\sum_{n|m} \mu(n) = 1(m = 1)$$

$$\sum_{n|m} \mu(n) = 0(m \neq 1)$$

proof. The case $m = 1$, $\sum_{n|m} \mu(n) = \mu(1) = 1$ is clear. $m \neq 1$, We factorise $m = {p_1}^{n_1}{p_2}^{n_2}{p_3}^{n_3}\cdots{p_k}^{n_k}$. We ignore zero term, $\sum_{n|m} \mu(n) = {}_k\mathrm{C}_0 - {}_k\mathrm{C}_1 + {}_k\mathrm{C}_2 - {}_k\mathrm{C}_3 + \cdots {}_k\mathrm{C}_k = (1-1)^k = 0$. □

[] is the Gauss sign.

Theorem 2.1.

$$\sum_{n \le m} \mu(n)\left[\frac{m}{n}\right] = 1$$

proof. By lemma2.1,$\sum_{m'=1}^{m}\sum_{n|m'} \mu(n) = 1$.

$$\sum_{m'=1}^{m}\sum_{n|m'} \mu(n) = (\mu(1))+(\mu(1)+\mu(2))+(\mu(1)+\mu(3))+(\mu(1)+\mu(2)+\mu(4))+\cdots$$

In this formula, we watch $\mu(n)$ as a character. $\mu(1)$ appears m times. $\mu(2)$ appears the number of the numbers of multiple of 2 lower than m that is $\left[\frac{m}{2}\right]$ times. $\mu(3)$ appears the number of the numbers of multiple of 3 lower than m that is $\left[\frac{m}{3}\right]$ times. $\mu(4)$ appears the number of the numbers of multiple of 4 lower than m that is $\left[\frac{m}{4}\right]$ times. Generally,in this formula, $\mu(n)(n \le m)$

appears the number of the numbers of multiple of n lower than m that is $\left[\frac{m}{n}\right]$ times. So we get

$$1=\sum_{m'=1}^{m}\sum_{n|m'}\mu(n)=(\mu(1))+(\mu(1)+\mu(2))+(\mu(1)+\mu(3))+(\mu(1)+\mu(2)+\mu(4))+$$

$$\cdots=\sum_{n\leq m}\mu(n)\left[\frac{m}{n}\right]$$

□

Theorem 2.2.

$$\sum_{n\leq m}\mu(n)=O(\sqrt{m}\log(m))$$

proof. First,we use induction method.We assume "For $m_0<M<m$,the absolute value of $\sum_{1\leq n\leq M}\mu(n)$ is less than constant K-multiple of $\sqrt{M}\times\log\sqrt{M}$". We take m_0 is big enohgh.
By Theorem2.1,

$$\sum_{n\leq\sqrt{m}}\mu(n)\left[\frac{m}{n}\right]+\sum_{\sqrt{m}<n\leq m}\mu(n)\left[\frac{m}{n}\right]=1$$

If we use induction method for $\sqrt{m}$. Next 2 formula is got.

$$\left|\sum_{n\leq\alpha_0}\mu(n)\left[\frac{m}{n}\right]\right|<\sqrt{m}m^{\frac{1}{4}}\log m^{\frac{1}{4}}\tag{1}$$

$$\left|\sum_{\alpha_0<n\leq m}\mu(n)\left[\frac{m}{n}\right]\right|<(\sqrt{m}\times m^{\frac{1}{4}}\log m^{\frac{1}{4}}-1)\tag{2}$$

We do not use (1), we use

$$\left|\sum_{n\leq\alpha_0}\mu(n)\right|<\sqrt{m}$$

$$\sum_{\alpha_0<n\leq m}\mu(n)\left[\frac{m}{n}\right]=(\left[\sqrt{m}\right]-1)\times\sum_{\alpha_0<n\leq m/\sqrt{m}-1}\mu(n)+(\left[\sqrt{m}\right]-2)\times$$

$$\sum_{m/\sqrt{m}-1<n\leq m/\sqrt{m}-2}\mu(n)+\cdots+1\times\sum_{m/2<n\leq m}\mu(n)$$

We take i maximum value satisfies $\frac{m}{\sqrt{m}-i+1}<\frac{m}{\log^2\sqrt{m}}$.We calculate well,$K\times(\sqrt{m}-1)$ is larger than right side all term's order. plus term and minus term exist so we can do this. The element of plus term delete with coeficient either, and the element of minus term delete with coeficient either. But $(\sqrt{m}-1),(\sqrt{m}-2)\cdots,1$ must not change. And 1 and -1 use same time. By assumption,$\left|\sum_{\sqrt{m}<n\leq\frac{m}{m-i+1}}\mu(n)\right|$'s absolute value is less than $K\times\sqrt{\frac{m}{m-i+1}}\log\sqrt{\frac{m}{m-i+1}}<K\times\sqrt{m}$.First term to i term is less than $K\times\sqrt{m}$. For correspond i term to last term, By $\sqrt{m}\times Km^{\frac{1}{4}}\log m^{\frac{1}{4}}-1>|\sum_{\sqrt{m}\leq n\leq\frac{m}{\sqrt{m}-i+1}}\mu(n)\left[\frac{m}{n}\right]+\sum_{\frac{m}{\sqrt{m}-i+1}<n\leq m}\mu(n)\left[\frac{m}{n}\right]|$, so less than $K\times\sqrt{m}$,too. Finally, left side large term move to right side.
All terms is less than $K\times\sqrt{m}$.
First induction is correct by later calculation.The statement is "For $m_0<M\leq m$(specially $M=m$),the absolute value of $\sum_{1\leq n\leq M}\mu(n)$ is less than constant K-multiple of $\sqrt{M}\times\log\sqrt{M}$". Next order's property is important.If $f(x),g(x)$ is same sign, then

$$O(f(x)+g(x))=O(max|f(x)|,|g(x|)$$

$$([\sqrt{m}]-1)\times\sum\mu(n)+([\sqrt{m}]-2)\times\sum\mu(n)+\cdots+1\times\sum\mu(n)$$

This formula is right side of formula that already done to delete. First we calculate

$$([\sqrt{m}]-1)\times\sum\mu(n)$$

(This term may be 0) $\sum\mu(n)$ has smaller order than $K\times(\sqrt{m}-1)$ times $\frac{1}{([\sqrt{m}]-1)}$ Repeat similler argument, by $\log\sqrt{m}\approx\frac{1}{([\sqrt{m}])}+\frac{1}{([\sqrt{m}]-1)}+\cdots+1$

$$\left|\sum_{\sqrt{m}<n\leq m}\mu(n)\right|<K((\sqrt{m}-1)\times\log\sqrt{m})$$

(Here,We only caluculate one of plus term's sum and minus term's sum)
Induction method is proved.

$$\sum_{n\leq m}\mu(n)=O(\sum_{n\leq\sqrt{m}}\mu(n)+\sum_{\sqrt{m}<n\leq m}\mu(n))=O(\sqrt{m}\log(\sqrt{m}))=O(\sqrt{m}\log(m))$$

□

Rifferences

[1]Riemann's zeta function,H.M.Edwards,Dover Books on Mathematics (28 Mar 2003)
[2]Uber die Anzahl der Primzahim unter einer gegebenen Grosse, Riemann. B, Monatsberichie der Berliner Akademie, November 1859.